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Technische Universität Dresden

Turbo Receivers for Equalizing Frequency-Selective MIMO Channels: Algorithms and Implementations

Tobias Seifert

von der Fakultät Elektrotechnik und Informationstechnik
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zur Erlangung des akademischen Grades

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Abstract

Received signals of modern wireless communications systems are impaired by temporal and spatial interferences between the symbols. Consequently, a sophisticated equalizer is required as an essential component of the receiver. When *a priori* knowledge based on decoder results is taken into account in the equalization procedure, interference can be reduced more efficiently than in conventional low-complex receivers. These schemes are referred to as *turbo equalization*.

This work focuses on turbo receiver schemes to equalize frequency-selective MIMO channels, which introduces intersymbol interference (ISI) and inter-antenna interference (IAI). The thesis is divided into two parts: The first part (Chapter 4) investigates different MMSE-based algorithms in terms of communications performance and computational complexity. In this context, a novel multi-iterative turbo receiver is proposed to enable interference-aware receiving with respect to the given channel scenario. It is furthermore shown, that normalizing the filter coefficients reduces the equalization bias and improves significantly the reliability of the transmission. The second part (Chapter 5) demonstrates an implementation of the developed algorithm based on a programmable solution. The equalizer application has been profiled whereas matrix inversion was identified as the most computational intensive operation. Starting from a RISC processor instruction set architecture, new equalizer-specific functional units (FUs) have been developed and integrated in order to accelerate both non-recursive and recursive operations. The design was optimized towards high area-efficiency, thus increasing the performance throughput with reasonable low additional hardware effort. In contrast to ASIC implementations, the resulting ASIP core enables very high flexibility to support different equalizer modes along with moderate to high performance.

Kurzfassung

Wichtiger Bestandteil gegenwärtiger und zukünftiger Mobilfunksysteme ist die Verwendung mehrerer Sende- und Empfangsantennen, die auf Empfangsseite zu räumlicher Überlagerung der Sendesignale führen. In Verbindung mit Mehrwegeausbreitung sind diese Signale zusätzlich von sogenannter zeitlicher Intersymbol-Interferenz geprägt. Zur Auflösung der Interferenzen benötigt es einen Entzerrer als zentrale Verarbeitungskomponente. Die vorliegende Arbeit beschäftigt sich mit iterativen Empfängerstrukturen, auch Turbo-Empfänger genannt, die einen signifikanten Leistungsgewinn der Übertragungszuverlässigkeit ermöglichen. Dabei werden neuartige, mehrfach-iterative Konzepte vorgestellt und sowohl hinsichtlich ihrer algorithmischen Komplexität als auch ihrer Leistungsfähigkeit untersucht. Neben algorithmischen Betrachtungen stehen insbesondere effiziente Implementierungen auf Basis einer erweiterten Befehlssatzarchitektur zur Beschleunigung der Kernoperationen des Entzerrers im Fokus dieser Arbeit.

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Finally, I also thank my parents for their immense trust and appreciation for everything I do.

Dresden, May 2017 Tobias Seifert

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Abbreviations

3GPP	3rd Generation Partnership Project
4G	4th generation of mobile communication technology
5G	5th generation of mobile communication technology
ADC	analog-to-digital converter
ADPLL	all-digital phase-locked loop
ALU	arithmetic logic unit
APP	*a posteriori* probability
ASIC	application-specific integrated circuit
ASIP	application-specific instruction-set processor
ATE	area-timing-energy
AWGN	Additive white Gaussian noise
BER	bit error rate
BPSK	binary phase shift keying
CDMA	code-division multiple access
CFO	carrier frequency offset
CGRA	coarse-grained reconfigurable architecture
CMOS	complementary metal-oxide-semiconductor
CoMP	coordinated multi-point
CP	cyclic prefix

CQI	channel quality indicator
CRC	cyclic redundancy check
CSI	channel state information
DFE	decision feedback equalizer
DFT	discrete Fourier transform
DPU	data plane processing unit
DSP	digital signal processor
EPA	extended pedestrian A
ETU	extended typical urban
EVA	extended vehicular A
EXIT	extrinsic information transfer
FB	feedback
FDE	frequency-domain equalizer
FDMA	frequency-division multiple access
FEC	forward error correction
FER	frame error rate
FFT	fast Fourier transform
FLIX	flexible-length instruction extension
FPGA	field-programmable gate array
FU	functional unit
GE	gate equivalents
GFDM	generalized frequency-division multiplexing
GPP	general-purpose processor
i.i.d.	independent and identically distributed
IAI	inter-antenna interference
IC	integrated circuit

IDE	integrated development environment
IDFT	inverse DFT
IEEE	Institute of Electrical and Electronics Engineers
IFFT	inverse FFT
ICI	intercarrier interference
IoT	internet of things
ISA	instruction set architecture
ISI	intersymbol interference
ITU	International Telecommunication Union
LLR	log-likelihood ratio
LTE	Long-Term Evolution
LTE-A	LTE Advanced
LUT	lookup table
LX5	Tensilica Xtensa LX5
MAP	maximum *a posteriori* probability
MGS	modified Gram-Schmidt
MIMO	multiple-input multiple-output
MLD	maximum likelihood detection
MLSE	maximum likelihood sequence estimation
MMSE	minimum mean square error
MPSoC	multiprocessor system-on-a-chip
NoC	Network-on-Chip
NWF	noise whitening filter
OFDM	orthogonal frequency-division multiplexing
OFDMA	orthogonal frequency-division multiple access
PA	power amplifier

PAPR peak-to-average-power ratio

PDF probability density function

PDP power delay profile

PE processing element

PIC parallel interference cancellation

PM processing module

QAM quadrature amplitude modulation

RB resource block

RISC reduced instruction set computing

rms root mean square

ROM read-only memory

RTL register-transfer level

SC-FDMA single-carrier frequency-division multiple access

SCM subcarrier mapping

SD sphere detector

SDK software development kit

SDR software-defined radio

SIC successive interference cancellation

SIMD single instruction, multiple data

SIMO single-input multiple-output

SINR signal-to-interference-plus-noise ratio

SISO soft-input soft-output

SLP Super Low Power

SNR signal-to-noise ratio

SO soft-output

STA synchronous transfer architecture

T4 Tomahawk4

TDMA time-division multiple access

TIE Tensilica instruction extension

TSD tuple-based SD

TSMC Taiwan Semiconductor Manufacturing Company

VLIW very large instruction word

WiMAX Worldwide Interoperability for Microwave Access

WLAN wireless local area network

ZF zero forcing

Notation and Symbols

Operators and Functions

$\mathbb{C}^{n\times m}$	set of complex numbers with dimensions $n \times m$
$\mathbb{R}^{n\times m}$	set of real numbers with dimensions $n \times m$
$\approx$	approximately equal
$\in$	element of
$\subseteq$	subset
$\sum$	summation of values
$\prod$	product of values
$\lvert \cdot \rvert$	absolute value of $(\cdot)$
$\lVert \cdot \rVert$	l^2-norm (Euclidean norm)
$(\cdot)^{\mathrm{T}}$	transpose of a matrix
$(\cdot)^{\mathrm{H}}$	conjugate transpose of a matrix (hermitian)
$(\cdot)^{-1}$	inverse of a matrix
$(\cdot)_{(\cdot)_t=0}$	sets t-th entry of a vector to zero
$(\cdot)_{\mathrm{diag}(\cdot)=0}$	sets the diagonal entries of a matrix to zero
$\otimes$	Kronecker product
$\arg\max_{\mathbf{x}}(\cdot)$	argument of the maximum of $(\cdot)$ over the set of points $\mathbf{x}$
$\arg\min_{\mathbf{x}}(\cdot)$	argument of the minimum of $(\cdot)$ over the set of points $\mathbf{x}$
$\mathcal{CN}(0, \sigma_{\mathrm{n}}^2)$	zero-mean complex-valued normal distribution of variance σ_{n}^2
$\mathrm{Cov}[\cdot]$	covariance matrix of a vector with random variable
$\mathrm{E}[\cdot]$	expected value of a random variable
$\mathrm{L}(\cdot)$	log-likelihood ratio values of an estimated symbol $(\cdot)$
$\mathrm{P}(\cdot)$	probability of event $(\cdot)$
$\mathrm{Var}[\cdot]$	variance of a random variable
$\mathrm{Im}\{\cdot\}$	imaginary part of $\{\cdot\}$

$\max_{\mathbf{x}}\{\cdot\}$	maximum of $\{\cdot\}$ over the set of points $\mathbf{x}$
$\min_{\mathbf{x}}\{\cdot\}$	minimum of $\{\cdot\}$ over the set of points$\mathbf{x}$
$\mathrm{Re}\{\cdot\}$	real part of $\{\cdot\}$

Symbols

$\mathbf{0}_{n\times m}$	zero or null matrix
$\mathbf{1}_T$	all-ones vector of length T
a_i	channel attenuation of path i
A	area of an integrated circuit
$\mathbf{A}$	intermediate matrix used for matrix inversion
B_{coh}	coherence bandwidth
B_{s}	signal bandwidth
$\underline{\mathbf{b}}$	time-domain feedback signal vector
$\underline{\mathbf{B}}$	frequency-domain feedback signal vector
$\underline{\mathbf{B}}_{\text{ISI}}$	portion of $\underline{\mathbf{B}}$ related to ISI
$\underline{\mathbf{B}}_{\text{IAI}}$	portion of $\underline{\mathbf{B}}$ related to IAI
$\underline{\mathbf{c}}$	time-domain feedforward filter block matrix
$\underline{\mathbf{C}}$	frequency-domain feedforward filter block matrix
$\mathbf{D}_t$	subcarrier mapping matrix of t-th transmit antenna
E	energy dissipated by an integrated circuit
f	clock frequency
$\mathbf{F}_N$	N-point Fourier matrix
$\mathbf{G}$	invertible $T \times T$ matrix
$h(t)$	continuous-time impulse response
$h[n]$	discrete-time impulse response
$h_{\text{ch}}(t)$	$h(t)$ of the propagation channel
$\vec{\underline{\mathbf{h}}}_{r,t}$	vector of channel impulse response between t and r
$\underline{\mathbf{h}}'_{r,t}$	time-domain channel matrix between t and r
$\underline{\mathbf{h}}_{r,t}$	effective $\underline{\mathbf{h}}'_{r,t}$, excludes non-occupied subcarriers
$\underline{\mathbf{H}}'_{r,t}$	frequency-domain channel matrix between t and r
$\underline{\mathbf{H}}_{r,t}$	effective $\underline{\mathbf{H}}'_{r,t}$, excludes non-occupied subcarriers
$\mathbf{I}_N$	identity matrix with dimensions $N \times N$
I	number of bits representing the integer part
J	cost function of the mean square error
K	number of subcarrier subsets with $K = M/S$
l	iteration level of the (multi-iterative) search tree
L	memory length of time-dispersive channel
$\hat{L}_j$	reciprocal values of diagonal elements $L_{j,j}$ in $\mathbf{L}$
$\mathbf{L}$	lower triangular matrix of Cholesky decomposition
m	time slot or subcarrier ($m \in \{1, 2, ..., M\}$)
M	number of occupied subcarriers
N	number of available subcarriers
$\underline{\mathbf{n}}'_r$	time-domain noise vector at r-th receive antenna

$\underline{\mathbf{n}}_r$	effective $\underline{\mathbf{n}}'_r$, excludes non-occupied subcarriers
$\underline{\mathbf{N}}'_r$	frequency-domain noise vector at r-th receive antenna
$\underline{\mathbf{N}}_r$	effective $\underline{\mathbf{N}}'_r$, excludes non-occupied subcarriers
P	number of pipeline stages per functional unit
r	receive layer ($r \in \{1, 2, ..., R\}$)
R	number of receiving antennas
S	number of grouped subcarriers (subcarrier set) with $S \leq M$
t	transmit layer ($t \in \{1, 2, ..., T\}$)
T	number of transmitting antennas
T_d	delay spread of time-dispersive channel
T_s	symbol duration of a signal
$\mathbf{u}$	source signal vector
$\mathbf{v}$	encoded $\mathbf{u}$
V_DD	power supply voltage
$\underline{\mathbf{w}}$	time-domain NWF matrix
W	bit-width for fixed-point format (integer and fractional part)
$x(t)$	transmit signal
x_t	constellation symbol sent by t-th transmit antenna
$\hat{x}_t$	(hard) estimation of the symbol sent by t-th transmit antenna
$\tilde{x}_t$	equalized (soft) symbol sent by t-th transmit antenna
$\underline{\mathbf{x}}'_t$	time-domain transmit symbol vector sent by t-th antenna
$\underline{\mathbf{x}}_t$	effective $\underline{\mathbf{x}}'_t$, excludes non-occupied subcarriers
$\underline{\mathbf{X}}'_t$	frequency-domain transmit symbol vector sent by t-th antenna
$\underline{\mathbf{X}}_t$	effective $\underline{\mathbf{X}}'_t$, excludes non-occupied subcarriers
$y(t)$	received signal
$\underline{\tilde{\mathbf{Y}}}_t$	IAI reduced frequency-domain received signal vector for t-th transmit antenna
$\underline{\mathbf{y}}'_r$	time-domain received symbol vector at r-th antenna
$\underline{\mathbf{y}}_r$	effective $\underline{\mathbf{y}}'_r$, excludes non-occupied subcarriers
$\underline{\mathbf{Y}}'_r$	frequency-domain received symbol vector at r-th antenna
$\underline{\mathbf{Y}}_r$	effective $\underline{\mathbf{Y}}'_r$, excludes non-occupied subcarriers

$\delta(t)$	Dirac delta function
η_i	relative number of the i-th receiver run (range $[0, 1]$)
μ	number of bits per constellation symbol (2^μ-QAM modulation order)
σ_{n}^2	noise variance
σ_{x}^2	average transmit power per antenna (equal for all antennas)
$\sigma_{\hat{\mathrm{x}},t}^2$	mean variance of the estimated symbols of the t-th antenna
τ_i	propagation delay of path i or latency of node i
τ_{max}	maximum latency constraint
$\bar{\tau}$	(average) equalization throughput
ρ	code rate
$\mathbf{\Gamma}_{TM}$	Kronecker product of $\mathbf{I}_T$ and $\mathbf{F}_M$
Δf	subcarrier spacing
$\mathcal{V}$	set of binary codewords $\underline{\mathbf{v}}$
$\mathcal{X}$	set of constellation symbols $\underline{\mathbf{x}}$

1 Introduction

Within less than two decades, mobile communication has changed our everyday life like only few other technical innovations before. While in the beginning, digital cellular networks focused on transmission of voice, mobile devices nowadays are particularly used for multimedia applications and data-intensive communication such as video streaming. Even though approximately half of the traffic is offloaded to WLAN fixed networks, the world's mobile data traffic is expected to increase in general by nearly 8-fold between 2015 and 2020 [Cis16]. By then, $75\,\%$ of that traffic will be video, which demands for increasing requirements on the performance of modern mobile communication systems. Since spectrum is a highly-limited resource, sophisticated transmissions techniques like multiple-input multiple-output (MIMO) antennas have been introduced in modern wireless communication standards such as IEEE 802.16 (WiMAX), 3GPP LTE or LTE Advanced [Ahm14]. MIMO is a techniques for sending multiple data streams simultaneously over the same radio channel. Ideally, it scales the channel capacity linearly with the number of transmitting antennas. Due to signal coupling at the receiver side, the computation effort for detecting the signal streams has drastically increased.

Besides throughput and transmission reliability, which can be improved by MIMO techniques as well, energy-efficiency has become of utmost importance. This demand is not only driven by cell phone users, but by emerging application trends and markets like internet of things (IoT), where different types of wireless sensor devices are connected to the mobile internet. Especially in the uplink, where data is transmitted from the battery-driven mobile device to the base station, it is important to reduce the transmit signal power. Also the dynamic range of the signal should be small to allow the power amplifier to operate more efficiently and to reduce the overall energy consumption. Regarding this, single-carrier modulation leads to much lower peak-to-average-power ratios than multi-carrier modulation[1], which is why it was deployed in the 3GPP LTE/LTE-A uplink. However, multipath propagation in combination with single-carrier transmission creates temporal

[1] Multi-carrier modulation mostly refers to orthogonal frequency-division multiplexing (OFDM), which is a spectrally-efficient method to allocate data on multiple carrier frequencies.

coupling between the symbols at the receiver side. This is also known as intersymbol interference (ISI). The larger the bandwidth[2], the stronger the coupling, which makes especially wideband transmission very susceptible to ISI. Furthermore, possible future transmission scheme candidates, such as generalized frequency-division multiplexing (GFDM), do no longer guarantee orthogonality between subcarriers. As a consequence, another type of interference, referred to as intercarrier interference (ICI), might be considered in the receiving processing.

To tackle all these kinds of occurring interferences, modern wireless receivers hence require an *equalizer* as essential component. The designs for such equalizers are usually based on dedicated hardware to fulfill high throughput requirements. On the other hand, the variety of existing wireless standards impose high demands on the compatibility and flexibility of modern receivers. Consequently, to support these different standards with common hardware resources, application-specific but still programmable realizations have become more and more attractive.

1.1 Scope and Outline

In 1995, Douillard and Berrou proposed an iterative procedure for correcting ISI [Dou+95]. Since this procedure was the direct application of turbo codes to an equalizer, it was then called *turbo equalization*. At the same time, it could be demonstrated [SKJ94; Czy97] that single-carrier modulation provides a performance-complexity trade-off similar to multi-carrier OFDM by making use of linear frequency-domain equalization (FDE). In the following years, FDE has hence become a promising equalization approach also in non-linear iterative schemes [BT05; SAS06]. When MIMO techniques came up, new sophisticated time-domain detection methods like tree-search detection [SB10] were developed to effectively benefit from spatial diversity. However, these concepts, even though combined with iterative detection-decoding, have mostly not taken ISI into account but assumed MIMO-OFDM channels only.

This work tries to overcome the separated views and hence investigate communications systems characterized by encoded data transmission over frequency-selective MIMO channels. The focus of the thesis is on receiver

[2] In LTE-A, subcarrier aggregation enables up to 100 MHz bandwidth [Ahm14].

schemes that generate very reliable estimation results or provide high communication performance. Regarding this, turbo equalization is considered to be the key technique towards this performance. The question arises about how to apply turbo equalization to MIMO systems that suffer from inter-antenna interference (IAI) and ISI? Furthermore, is it still possible and beneficial to use sophisticated MIMO detection methods at feasible complexity? The work investigates the considered schemes from both an algorithmic and implementation perspective. It hence also covers design issues that arise when aiming for a high-performing *and* programmable solution. The remainder of the thesis is organized as follows:

- ▷ **Chapter 2** provides background information about the phenomenon of classical multipath propagation and how to model the frequency-selective channels. The transmissions schemes for 3GPP LTE/LTE Advanced are introduced, especially with focus on single-carrier transmission in the uplink. The chapter concludes with a detailed analysis of the channel impulse responses, taking different power delay profiles and bandwidth assumptions into account.

- ▷ **Chapter 3** introduces the encoded communications system model for a generic MIMO setup. An overview about existing equalization and detection methods for those systems is presented, ranging from linear equalization to optimal maximum *a posteriori* probability (MAP) decoding. The concept of turbo receiving to approximate MAP receivers is also introduced.

- ▷ **Chapter 4** extends turbo equalization to MIMO transmission. The channel equalization is performed jointly in the frequency-domain. Subsequently, a multi-iterative receiver scheme, that comprises two stages to specifically address different types of interference, is proposed. It is further demonstrated that normalization of the equalization filter tremendously enhances the communications performance. The chapter concludes with a detailed performance-complexity investigation for different MIMO setups.

- ▷ **Chapter 5** focuses on the hardware implementation of the selected soft-input soft-output FDE running on an application-specific instruction-set processor (ASIP). For this purpose, the equalization algorithm is firstly prepared for low-cost implementation. The subsequent part develops additional hardware features that can be integrated for accelerating the execution. Besides synthesis results, measurement results of a silicon implementation of the ASIP are finally presented.

▷ **Chapter 6** summarizes the key results of the thesis and gives an outlook about future research topics.

1.2 Notation

The following section gives an overview about important notation used within the thesis. An important remark concerns the distinction between frequency-domain and time-domain values, which is essential to improve clarity of formulas. Any further notations can be looked up in the detailed notation list, starting on page xxv.

- Unless otherwise stated, the used variables x are generally complex-valued, i.e., $x \in \mathbb{C}$.

- Italic characters (e.g., x, X) indicate scalar variables or constants. Bold characters (e.g., $\mathbf{x}$, $\mathbf{X}$) always represent vectors or matrices. Hereby, x_i identifies the i-th element of the vector. If $\mathbf{x}$ is a matrix, x_{jk} identifies the element placed in the j-th row and k-th column.

- Vectors and matrices that refer to an M-length SC-FDMA symbol are additionally underlined, e.g., $\underline{\mathbf{x}}$. The dimension length of those vectors and matrices is always an integer multiple of M.

- SC-FDMA related vectors and matrices are represented either in time- or frequency-domain. Lower case vectors or matrices (e.g., $\underline{\mathbf{x}}$, $\underline{\mathbf{h}}$) indicate time-domain representation. Upper case vectors or matrices (e.g., $\underline{\mathbf{X}}$, $\underline{\mathbf{H}}$) indicate frequency-domain representation.

- SC-FDMA related vectors are often stacked subvectors. While $\underline{\mathbf{x}}_j$ addresses the j-th subvector, $\mathbf{x}[m]$ composes a vector based on the m-th elements of all subvectors. To address a scalar in the stacked vector, $x_j[m]$ is used. Note that the underline has been removed since the latter two examples do not refer to a complete SC-FDMA symbol anymore, but to a single time slot or subcarrier m.

- SC-FDMA related matrices are often block matrices. To address the submatrix in the j-th row and k-th column, the notation the $\underline{\mathbf{H}}_{j,k}$ is used. To extract a matrix composed of the m-th diagonal element of all submatrices, the notation $\mathbf{H}[m]$ is used.
- $\mathbf{H}^{\mathrm{H}}$ and $\mathbf{H}^{\mathrm{T}}$ denote the hermitian and transpose of a matrix/vector $\mathbf{H}$, respectively. $\mathbf{H}^{-1}$ denotes the inverse of the squared matrix $\mathbf{H}$.
- $\mathbf{F}_M$ and $\mathbf{F}_M^{\mathrm{H}}$ are the Fourier matrix of an M-point DFT and the inverse Fourier matrix of an M-point IDFT, respectively.